On the behaviour of small disturbances in plane Couette flow

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The problem considered here is concerned with small disturbances of plane Couette flow. As is usual in such problems it is assumed that the disturbance velocities are sufficiently small to allow the Navier–Stokes equations to be linearized. There results a special case of the well-known Orr–Sommerfeld equation and this is solved by an exact method using a digital computer. The problem has previously been considered by several authors, mostly using approximate methods and their results have been compared where possible with those obtained here. It was possible to proceed to values of αR not in excess of 1000 (α being the wave-number of the disturbance and R the Reynolds number of the basic flow), and the results tend to confirm the belief that Couette flow is stable at all Reynolds numbers.

Introduction

This paper presents extensive numerical results concerning the behaviour of infinitesimal disturbances of plane Couette flow. We consider the fluid to be viscous and incompressible and assume that the main flow takes place in the direction of the x-axis. Having linearized the equations governing the disturbances we may assume that the disturbance velocity, for example in the transverse direction, is obtainable from a superposition of functions of the type $v(y)\exp\{i\alpha(x-\xi t)\}$. In this expression α is the wave-number which is considered to be real and positive, while the stability of the motion depends on ξ, which in turn depends on α and on the Reynolds number R of the basic flow. The admissible values of ξ must be calculated from the differential equation

$$v^{\mathrm{iv}} - 2\alpha^2 v'' + \alpha^4 v - i\alpha R(y-\xi)(v''-\alpha^2 v) = 0, \tag{1}$$

of which ξ is the eigenvalue.

Previously this problem has been considered by many authors, although few of these have attempted a numerical solution. Their conclusions all suggest that the motion is stable at all Reynolds numbers. An extensive investigation of this problem was undertaken by Hopf (1914), who expressed the solutions of the above equation in terms of integrals involving Bessel functions. He found it necessary to approximate these functions by their asymptotic developments and as a result his calculations are limited to the case of small and very large αR. It is found that if αR is sufficiently small the eigenvalues ξ are purely imaginary and for this range of αR another investigation was carried out by Southwell & Chitty (1930).

More recently the case of $\alpha = 1$ was extensively treated by Grohne (1954) by an approximate analysis. (For general information, see also Wasow 1953.)

We have employed in the present paper a method of solution which is not unusual (at least where differential equations with real coefficients are concerned) but which can be very powerful when coupled with the use of a digital computer. It is, briefly, to replace the fourth-order differential equation by a set of algebraic equations whose solutions are the coefficients of an orthogonal expansion. Using this method we have been able to proceed to values of αR well beyond those for which ξ ceases to be purely imaginary, thus obtaining information for values of this parameter in a range previously considered only by Grohne in the case of $\alpha = 1$. Furthermore, it should be pointed out that it was not necessary to make any approximations beyond that of truncating the matrices at the final stage. Apart from the interest of the present problem itself, extensive solutions were required as a basis for solving the more general problem in which the lower plate is at a higher temperature than the upper one. The results of this later problem will be published shortly. As shall be seen below it turns out that although the basic differential equation has complex coefficients the analysis can be framed to provide a set of algebraic equations whose matrix is composed entirely of real elements. This is a considerable simplification which makes the problem ideally suited to solution on an automatic digital computer.

1. Formulation of the problem

Since the original three-dimensional problem can be reduced to a two-dimensional one with a lower Reynolds number (see Lin 1955, p. 27), we take axes Ox' and Oy' in the plane of flow, the two plates forming the lines $y' = \pm l$. These plates we consider to move with equal and opposite velocities U_0 parallel to the x'-axis. We may now express the problem in non-dimensional form by taking U_0 and l as representative of velocity and length respectively. The basic velocity field is now given by

$$\bar{u} = y, \quad \bar{v} = 0 \quad (-1 \leqslant y \leqslant +1).$$

If the disturbed velocity field is described by

$$u = \bar{u} + \hat{u}(x,y,t), \quad v = \bar{v} + \hat{v}(x,y,t),$$

then on substituting these in the Navier–Stokes equations, neglecting second order quantities, and separating variables by the substitutions

$$\left.\begin{aligned} \hat{u} &= u(y)\exp\left[i\alpha(x-\xi t)\right], \\ \hat{v} &= v(y)\exp\left[i\alpha(x-\xi t)\right], \\ \hat{p} &= p(y)\exp\left[i\alpha(x-\xi t)\right], \end{aligned}\right\} \tag{1.1}$$

we are led to a particular case of the well-known Orr–Sommerfeld equation (Lin, 1955, p. 7).

$$\left[\frac{d^2}{dy^2} - \alpha^2 - i\alpha R(y-\xi)\right]\left[\frac{d^2}{dy^2} - \alpha^2\right] v = 0, \tag{1.2}$$

with boundary conditions

$$v = dv/dy = 0 \quad \text{at} \quad y = \pm 1, \quad \text{where} \quad R = U_0 l/\nu.$$

The criterion for stability is that $\operatorname{Im}\xi < 0$, while if $\operatorname{Im}\xi > 0$ the flow is unstable.

The form of equation (1.2) is that most common, but in the present case it is convenient to transform the range of integration by the substitution

$$y_1 = \tfrac{1}{2}\pi(y+1),$$

so that $0 \leqslant y_1 \leqslant \pi$. The equation (1.2) now reads

$$\left[\frac{d^2}{dy_1^2}+\lambda+iby_1\right]\left[\frac{d^2}{dy_1^2}+c\right]v = 0, \tag{1.3}$$

with $$v = dv/dy_1 = 0 \quad \text{at} \quad y_1 = 0 \quad \text{and} \quad y_1 = \pi,$$

where for convenience we use the notation

$$c = -4\alpha^2/\pi^2, \quad b = -8\alpha R/\pi^3, \quad \lambda = c - \tfrac{1}{2}\pi bi(1+\xi). \tag{1.4}$$

There will be no confusion if in what follows we write y for y_1.

2. The basic orthogonal functions

In general the solution v and the eigenvalue λ will be complex and the function $v(y)$ may be considered either as a function of a complex variable or as a complex function of the real variable y. If we adopt the latter interpretation we may write

$$v(y) = v_1(y) + iv_2(y) \quad (0 \leqslant y \leqslant \pi),$$

where v_1 and v_2 are real quantities. Now both v_1 and v_2 may be expressed as series of orthogonal functions (say $Y_r(y)$) so clearly we may put

$$v(y) = \sum_{r=1}^{\infty} a_r Y_r(y), \tag{2.1}$$

where the coefficients a_r are complex numbers.

The particular orthogonal functions which we use (see also Chandrasekhar & Reid 1957) are those defined (except for a multiplicative constant) by the differential system

$$\left.\begin{aligned} \frac{d^4Y}{dy^4} &= \mu^4 Y, \\ Y = dY/dy = 0 \quad \text{at} \quad y &= 0 \quad \text{and} \quad y = \pi. \end{aligned}\right\} \tag{2.2}$$

The orthogonality of these functions is easily verified so that we have

$$I_{rn}^{(0)} = \int_0^\pi Y_r Y_n \, dy = 0 \quad \text{when} \quad r \neq n.$$

We also introduce the abbreviations

$$I_{rn}^{(1)} = \int_0^\pi Y_r' Y_n \, dy, \quad I_{rn}^{(2)} = \int_0^\pi Y_r' Y_n' \, dy,$$

$$J_{rn}^{(0)} = \int_0^\pi y Y_r Y_n \, dy, \quad J_{rn}^{(2)} = \int_0^\pi y Y_r' Y_n' \, dy.$$

For further details of these functions we refer the reader to the Appendix.

3. Reduction of the differential equation to a set of algebraic equations

It is now our purpose to evaluate the eigenvalues of the differential equation (1.3). The method used is quite straightforward, being to replace equation (1.3) by an infinite set of linear algebraic equations containing an eigenvalue.

We multiply (1.3) throughout by $Y_n(y)$ and integrate over the interval $0 \leqslant y \leqslant \pi$. On integration by parts we have

$$\int_0^\pi v[Y_n^{\mathrm{iv}} + (\lambda + c + iby)\, Y_n'' + 2ib\, Y_n' + c(\lambda + iby)\, Y_n]\, dy = 0.$$

On substituting for v from (2.1), interchanging the orders of integration and summation, this becomes:

$$\sum_{r=1}^{\infty} [(\mu_n^4 + \lambda c)\, I_{nr}^{(0)} - (\lambda + c)\, I_{nr}^{(2)} + ib(I_{nr}^{(1)} - J_{nr}^{(2)}) + ibc J_{nr}^{(0)}]\, a_r = 0. \tag{3.1}$$

Having divided the nth member of this set of equations by μ_n^4 which ensures convergence of its determinant we may write (3.1) in the matrix form

$$[\mathbf{A}_1 + i\mathbf{A}_2 - \lambda' \mathbf{B}]\, \mathbf{a} = 0, \tag{3.2}$$

where $\lambda' = \lambda + \frac{1}{2}\pi b i$. λ' has been chosen so as to make the elements of $\mathbf{A}_2$ zero whenever $n + r$ is even. In this $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{B}$ are real matrices and are given by

$$\begin{aligned} \mathbf{A}_1 &= [\mu_n^4\, I_{nr}^{(0)} - c I_{nr}^{(2)}], \\ \mathbf{A}_2 &= [b(I_{nr}^{(1)} - J_{nr}^{(2)} + \tfrac{1}{2}\pi I_{nr}^{(2)}) + bc(J_{nr}^{(0)} - \tfrac{1}{2}\pi I_{nr}^{(0)})], \\ \mathbf{B} &= [I_{nr}^{(2)} - c I_{nr}^{(0)}]. \end{aligned}$$

We are now faced with the problem of determining the latent roots of a complex matrix. It appears, however, on closer inspection that this problem can be reduced to that of finding the latent roots of a real matrix. With reference to the Appendix we see that the matrices $\mathbf{A}_1$ and $\mathbf{B}$ have zeros in the positions were $n + r$ is odd and as we have arranged above, $\mathbf{A}_2$ in the positions where $n + r$ is even. If we define the matrices $\mathbf{T}$ and $\mathbf{S}$ as follows

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & i & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & i & \dots \\ . & . & . & . & \end{bmatrix}, \mathbf{S} = -\mathbf{T}^2,$$

then the transformation $$\mathbf{a} = \mathbf{T}\mathbf{d}$$

puts (3.2) into the form $$[\mathbf{A} - \lambda' \mathbf{B}]\, \mathbf{d} = 0, \tag{3.3}$$

where $$\mathbf{A} = \mathbf{A}_1 + \mathbf{S}\mathbf{A}_2.$$

The matrices $\mathbf{A}$ and $\mathbf{B}$ in (3.3) are real.

4. Calculations and conclusion

The infinite matrix equation (3.3) was approximated by its first N rows and columns. Once the basic matrices $[I^{(1)}_{nr}]$ etc. were computed it was a simple matter to vary the parameters b and c. A typical calculation for fixed values of b and c consisted of extracting the N latent roots of the abbreviated matrix. This calculation was repeated increasing the value of N until, to the prescribed accuracy, the first latent root was unchanged. The capacity of the computer used allowed N to be increased if necessary to the value $N = 20$. Although our primary interest was in the first latent root it was found that several higher roots were also accurately calculated.

It is interesting to notice how the usual difficulty of dealing with large values of αR made itself felt in the matrix equation. We write (3·3) in the form

$$[\mathbf{A}_1 + \mathbf{S}\mathbf{A}_2 - \lambda'\mathbf{B}]\,\mathbf{d} = 0. \tag{4.1}$$

In this equation the only matrix affected by increase of αR, or equivalently b, is the matrix $\mathbf{A}_2$, every element of which is multiplied by b. However large a value is prescribed for b the (nr)th element of this matrix will tend to zero as $n \to \infty$ and $r \to \infty$ but if we are restricted to a finite $N \times N$ approximation to (4.1) it is possible to choose b so large that the eventual dominance of the elements of $\mathbf{A}_1$ does not appear. Thus if b is indefinitely increased in the $N \times N$ matrix we shall be evaluating approximations to the latent roots, not of the equation (4.1) but of

$$[\mathbf{S}\mathbf{A}_2 - \lambda'\mathbf{B}]\,\mathbf{d} = 0.$$

Thus in analogy with the disappearance of the highest derivative in the Orr–Sommerfeld equation when $\alpha R \to \infty$ the matrix $\mathbf{A}_1$ disappears in our approximate matrix equation. The highest value of b that could be considered was that corresponding to $\alpha R = 1000$.

To the best of our knowledge this problem has not previously been the subject of an exact numerical investigation, except for the work of Southwell & Chitty mentioned above. However, extensive numerical approximations have been carried out by other authors and in this paragraph we compare our results with these. In one trivial case, namely when $\alpha = 0$ and R is finite, exact solutions of the differential equation (1.2) are easily obtained and here the agreement with the present calculations is exact. As was mentioned in the introduction the work of Hopf, Southwell & Chitty and Grohne provide three other sources of comparison. In table 1 our results are compared with those of Hopf. The error in the latter's work, due to the approximations used by him, is generally estimated to be of the order of 10 %. This is found to be true for all but one of the compared values. Referring to the results of Southwell & Chitty a comparison is available when αR is small enough to allow ξ to be purely imaginary. From (1.4) and (3.2) this means that λ' is purely real when αR is sufficiently small. The insert of figure 1 shows this comparison. There λ'_1 and λ'_2 denote the first and second eigenvalues of (4.1) respectively. Finally, in the case $\alpha = 1$, the value of αR at which ξ ceases to be purely imaginary was calculated by the present method to be 66. A graph given by Southwell & Chitty (p. 209) also gives a value of 66, while the two graphs given by Grohne give the two different values of 68 and 76.

The results of the present calculations for the first eigenvalue appear in tables 2 to 4. These are illustrated in terms of the eigenvalue ξ by figures 3 and 4.

Although we have been able to proceed to considerably higher values of αR than most previous authors, we cannot proceed indefinitely for the reason given above. Thus we are unable to say for certain that shear flow is always stable, that

αR	α	Re λ_1' (Hopf)	Re λ_1' (present)
0·125	0	3·99	4·00
	1	3·28	3·37
	2·5	2·22	2·19
1·25	0	3·94	4·00
	1·075	3·06	3·30
	3·225	1·88	1·90
12·5	0	3·20	3·93
	2·32	2·66	2·59
	4·64	2·07	2·11

TABLE 1

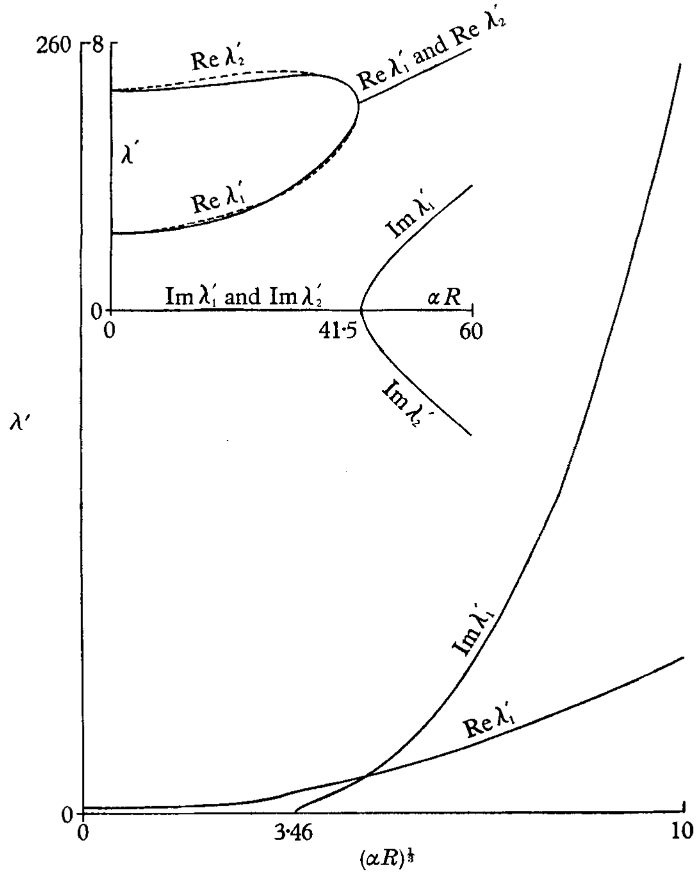

FIGURE 1. Variation of λ' with αR for the case $\alpha = 2$; – – –, values obtained by Southwell & Chitty.

is to say that $\operatorname{Im} \xi$ is always negative, but the trend of the results suggests this very strongly (see figure 3). Probably one of the most interesting aspects of the results obtained is the fact already mentioned that as αR is increased, keeping α fixed, the first eigenvalue is real and increasing for a while until it reaches a certain value when, coinciding with another (real) eigenvalue, these split into a

αR	$\alpha = 0$	$\alpha = 0{\cdot}5$	$\alpha = 1$	$\alpha = 1{\cdot}5$	$\alpha = 2$	$\alpha = 4$	$\alpha = 6$	$\alpha = 8$
1	4·00	3·81	3·37	2·89	2·50	1·70	1·43	1·31
10	3·95	3·79	3·42	3·00	2·66	1·99	1·82	1·77
20	3·87	3·79	3·57	3·34	3·16	3·08	3·71	3·87
30	3·89	3·89	3·90	3·96	4·13	5·11	4·79	4·66
40	4·11	4·20	4·48	4·98	6·18	5·98	5·74	5·65
50	4·62	4·82	5·44	6·83	7·82	6·97	6·71	6·60
60	5·56	5·91	7·16	9·18	8·75	7·93	7·63	7·48
70	7·31	8·12	10·0	9·84	9·59	8·84	8·49	8·31
80	10·2	10·3	10·4	10·5	10·4	9·71	9·31	9·11
90	10·4	10·6	10·8	11·1	11·2	10·5	10·1	9·88
100	10·8	10·9	11·3	11·8	12·0	11·4	10·9	10·6
300	21·0	—	—	—	24·4	24·6	23·5	22·8

TABLE 2. $\operatorname{Re} \lambda_1'$ tabulated against α and αR

αR	$\alpha = 0$	$\alpha = 0{\cdot}5$	$\alpha = 1$	$\alpha = 1{\cdot}5$	$\alpha = 2$	$\alpha = 4$	$\alpha = 6$	$\alpha = 8$
1	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0·956
30	0	0	0	0	0	1·66	2·47	2·86
40	0	0	0	0	0	3·34	4·21	4·70
50	0	0	0	0	2·41	5·05	6·13	6·77
60	0	0	0	2·52	4·18	6·93	8·19	8·94
70	0	0	2·56	4·55	5·93	8·91	10·3	11·2
80	2·80	3·46	4·85	6·34	7·69	11·0	12·6	13·5
90	5·18	5·62	6·70	8·07	9·47	13·1	14·8	15·9
100	7·04	7·45	8·45	9·82	11·3	15·2	17·2	18·3
300	49·6	—	—	—	56·3	64·7	69·4	72·2

TABLE 3. $\operatorname{Im} \lambda_1'$ tabulated against α and αR

αR	200	300	400	500	600	700	800	900	1000
$\operatorname{Re} \lambda_1'$	18·9	24·4	29·2	33·5	37·6	41·4	45·0	48·5	51·9
$\operatorname{Im} \lambda_1'$	32·4	56·3	82·0	109	136	165	194	223	253

TABLE 4. $\operatorname{Re} \lambda_1'$ and $\operatorname{Im} \lambda_1'$ tabulated against αR for the case $\alpha = 2$

complex conjugate pair. In the case of $\alpha = 2$ this behaviour is illustrated in figure 1, the first eigenvalue ceasing to be real when $\alpha R = 41{\cdot}5$. The physical interpretation, referring to figure 1 is that for $\alpha R < 41{\cdot}5$ a disturbance with wave number $\alpha = 2$ corresponding to this eigenvalue will remain stationary relative to a line midway between the two plates, whereas if $\alpha R > 41{\cdot}5$ there will be two disturbances one moving upstream, the other downstream, with equal velocities relative to this datum. This transition is quite abrupt and we felt it to be

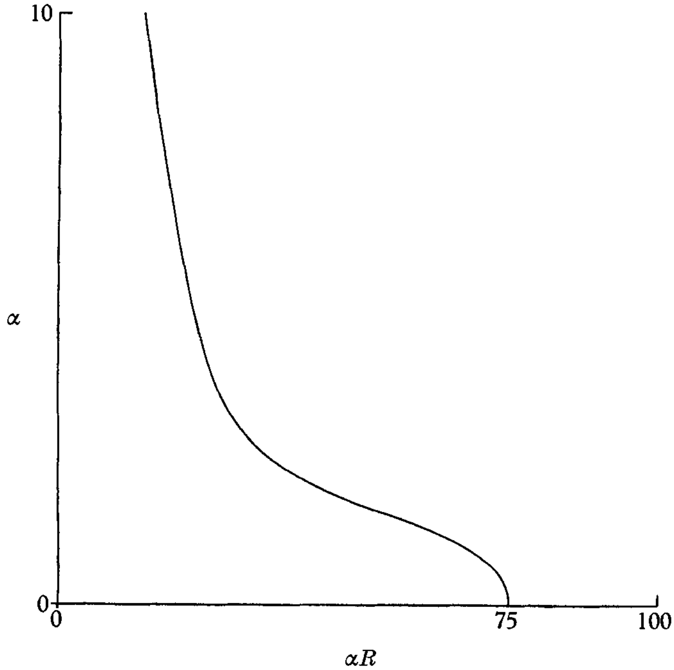

FIGURE 2. The graph of α against αR for which the first two eigenvalues coincide.

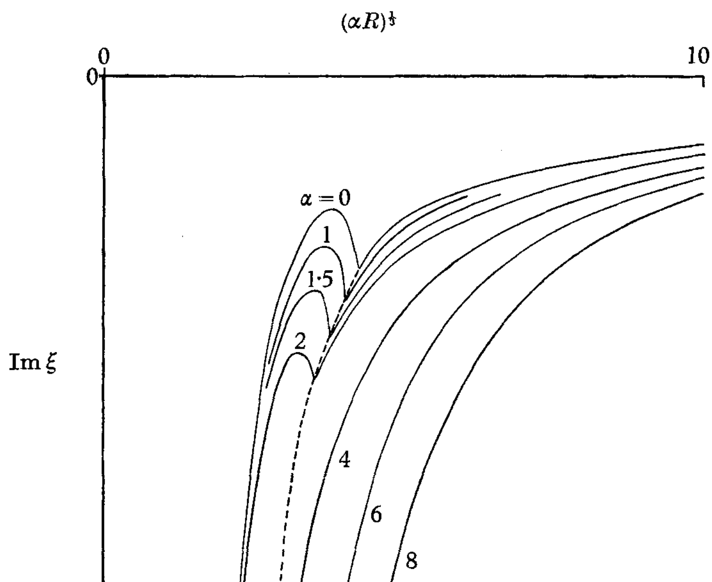

FIGURE 3. Variation of $\operatorname{Im}\xi$ with αR for various values of α. The first two eigenvalues coincide along the dotted line.

sufficiently interesting to include figure 2 where this critical value of αR is plotted against α. This transition curve is also included as the dotted line in figures 3 and 4.

In our notation Grohne concluded (as did Hopf) that as $\alpha R \to \infty$

$$\xi \to \pm 1.$$

Although the values of αR to which we have been able to proceed are limited, our calculations suggest that the first eigenvalue tends to $+1$ while the second eigenvalue tends to -1.

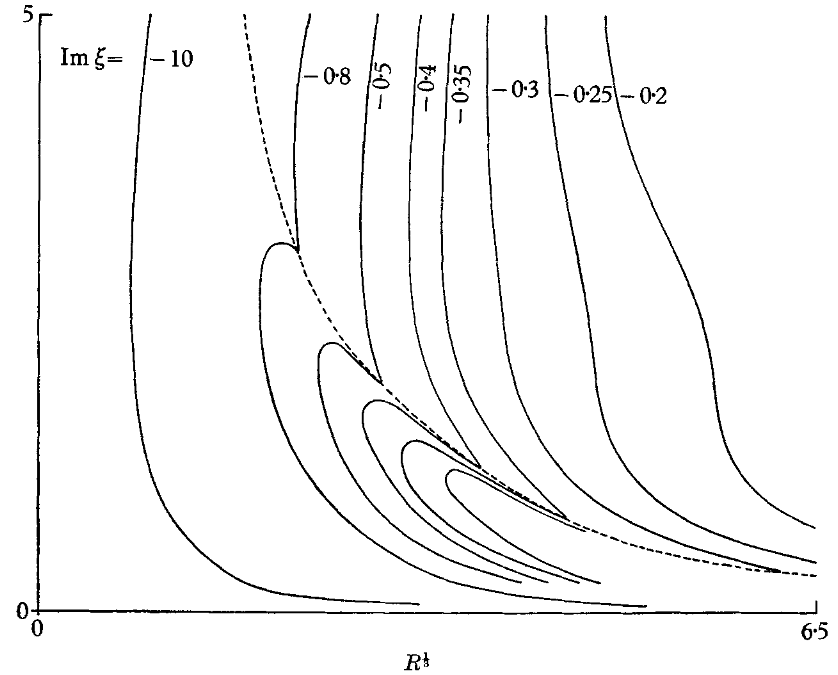

FIGURE 4. Contours of constant Im ξ. The first two eigenvalues coincide along the dotted line.

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Appendix

(i) *Basic orthogonal functions* $Y_r (r = 1, 2, 3, \ldots)$

The solutions of the differential system

$$Y^{\text{iv}} - \mu^4 Y = 0,$$

$$Y = dY/dy = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = \pi,$$

are
$$Y_r = (\cosh \mu_r y - \cos \mu_r y) - \beta_r (\sinh \mu_r y - \sin \mu_r y),$$

where
$$\beta_r = \frac{\cosh \mu_r \pi - \cos \mu_r \pi}{\sinh \mu_r \pi - \sin \mu_r \pi}, \quad \text{and} \quad \mu_r (r = 1, 2, 3, \ldots)$$

are the roots of the equation $\cosh \mu\pi \cos \mu\pi = 1$.

(ii) *Evaluation of integrals* $I^{(0)}_{rn}$ *etc.*

The value of the basic integrals is given by:

$$I^{(0)}_{rn} = \begin{cases} 0, & r \neq n, \\ \pi, & r = n; \end{cases}$$

$$I^{(1)}_{rn} = \begin{cases} 0, \text{ when } r+n \text{ is even,} \\ \dfrac{8\mu_r^2 \mu_n^2}{\mu_n^4 - \mu_r^4}, \text{ when } r+n \text{ is odd;} \end{cases}$$

$$I^{(2)}_{rn} = \begin{cases} \dfrac{-8\mu_r^2 \mu_n^2}{\mu_r^4 - \mu_n^4} (\mu_r \beta_r - \mu_n \beta_n), \text{ when } r+n \text{ is even, } r \neq n, \\ 0, \text{ when } r+n \text{ is odd,} \\ \pi \mu_n^2 \beta_n^2 - 2\mu_n \beta_n, \text{ when } r = n; \end{cases}$$

$$J^{(0)}_{rn} = \begin{cases} 0, \text{ when } r+n \text{ is even, } r \neq n, \\ \dfrac{-32\mu_r^3 \mu_n^3 \beta_r \beta_n}{(\mu_r^4 - \mu_n^4)^2}, \text{ when } r+n \text{ is odd,} \\ \pi^2/2, \text{ when } r = n; \end{cases}$$

$$J^{(2)}_{rn} = \begin{cases} \dfrac{-4\pi \mu_r^2 \mu_n^2}{\mu_r^4 - \mu_n^4} (\mu_r \beta_r - \mu_n \beta_n), \text{ when } r+n \text{ is even, } r \neq n, \\ \dfrac{4\pi \mu_r^2 \mu_n^2}{\mu_r^4 - \mu_n^4} (\mu_r \beta_r - \mu_n \beta_n) - \dfrac{16(\mu_r^4 + \mu_n^4)}{(\mu_r^4 - \mu_n^4)^2} \mu_r^2 \mu_n^2, \text{ when } r+n \text{ is odd,} \\ \tfrac{1}{2}\pi(\pi \mu_n^2 \beta_n^2 - 2\mu_n \beta_n), \text{ when } r = n. \end{cases}$$